Extra Practice for Induction

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CS1231S – T02A

Extra Questions:

1. Prove that
2. Prove that
3. Prove that the number of nonempty subsets of a set with elements is .
4. Prove that .
5. Prove that can be written uniquely as the product of prime numbers.
6. Alice drawn lines on a sheet of paper, where such that no two of them are parallel. The lines split up the sheet of paper into different regions. How many regions are there?
7. Alice and Bob plays a game. There are stones, where is a positive integer. In each turn, a player can take any number of stones between 1 and inclusive, where is a given positive integer . A player is considered to have lost if he or she cannot make a valid move at his or her turn. Alice starts first.
   1. For what values of can Alice guarantees a win?
   2. For what values of can Bob guarantees a win?
8. Let such that is an integer. Prove that is an integer.

Challenge Problems:

1. Let be a sequence of real numbers such that Prove that ,

(Hint: Use strong induction)

1. Let be two positive integers. Prove that there exist positive integers such that

(Hint: Try inducting over . Split cases where is odd or even)

Solutions and Hints to selected problems:

I will only give hints and key ideas for Extra Questions. It is meant to be for your practice, so try to fill in the gaps by yourselves.

1. Straight forward induction. Just notice that
2. Notice that

And

Split the problems into 2 subproblems.

1. Consider a set with elements. The -th element can either in or not in the subset generated by the remaining elements.
2. Notice that
3. Prove this using strong induction. Divide case where is prime and when is composite.
4. The answer is

The idea is to look at the fact that if you already have lines and you want to add the -th line, the new line intersects every other line because its not parallel to any other line. Hence, it adds an extra regions.

1. The key idea is that if a player has taken stones, where , then the other player can take stones on the next turn since . Using the key idea, prove that Alice can guarantee a win if and Bob can guarantee a win if .

Further hint, use strong induction on the predicate

The first player can guarantee a win if and the second player can guarantee a win if , .

1. Notice that

I will give full solutions for Challenge Problems.

1. Let .
   1. Notice that . Hence is true.
   2. Let Assume that are true. We have
   3. Add all those inequalities, we have
   4. Add to both sides. We have
   5. Notice that
   6. Hence,
   7. Hence
   8. Add both to both sides.
   9. is true.
   10. Hence, is true by Strong MI.
2. Let for all such that

,

* 1. If , then take . We have
  2. Hence, is true.
  3. Let Assume that is true.
  4. Case 1: is even.
  5. Then we can write , (by definition of even numbers)
  6. Notice that
  7. Hence, we can take . Notice that since and by closure of integers.
  8. By the induction hypothesis

For some positive integers .

* 1. Hence,
  2. Case 2: is odd.
  3. Then we can write + 1, (by definition of odd numbers)
  4. Notice that
  5. Hence, we can take . Notice that by closure of integers and .
  6. By the induction hypothesis

For some positive integers .

* 1. Hence,
  2. In all cases is true.
  3. Hence, is true by MI.